ABSOLUTE GAP-SHEAVES AND EXTENSIONS OF COHERENT ANALYTIC SHEAVES

BY

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Thimm introduced the concept of gap-sheaves for analytic subsheaves of finite direct sums of structure-sheaves on domains of complex number spaces (Definition 9, [13]) and proved that these gap-sheaves are coherent if the subsheaves themselves are coherent (Satz 3, [13]). This concept of gap-sheaves can be readily generalized to analytic subsheaves of arbitrary analytic sheaves on general complex spaces (Definition 1, [12]). All the gap-sheaves of coherent analytic subsheaves of arbitrary coherent analytic sheaves on general complex spaces are coherent (Theorem 3, [12]). The gap-sheaves of a given analytic subsheaf depend not only on the subsheaf itself but also on the analytic sheaf in which the given subsheaf is embedded as a subsheaf.

In this paper we introduce a new notion of gap-sheaves which we call absolute gap-sheaves (Definition 3 below). These gap-sheaves arise naturally from the problem of removing singularities of local sections of a coherent analytic sheaf. They depend only on a given analytic sheaf and neither require nor depend upon an embedding of the given sheaf as a subsheaf in another analytic sheaf. We give here a necessary and sufficient condition for the coherence of absolute gap-sheaves of coherent sheaves (Theorem 1 below). This yields some results concerning removing singularities of local sections of coherent sheaves (see Remark following Corollary 2 to Theorem 1). Then we use absolute gap-sheaves to derive a theorem (Theorem 2 below) which generalizes Serre's Theorem on the extension of torsion-free coherent analytic sheaves (Theorem 1, [11]). Finally a result on extensions of global sections of coherent analytic sheaves is derived (Theorem 4 below).

Unless specified otherwise, complex spaces are in the sense of Grauert (§1, [5]). If $\mathscr{S}$ is an analytic subsheaf of an analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$, then $\mathscr{S} : \mathscr{T}$ denotes the ideal-sheaf $\mathscr{I}$ defined by $\mathscr{I}_x=\{s \in \mathscr{H}_x \mid s\mathscr{T}_x \subset \mathscr{S}_x\}$ for $x \in X$. $E(\mathscr{S}, \mathscr{T})$ denotes $\{x \in X \mid \mathscr{S}_x \neq \mathscr{T}_x\}$. Supp $\mathscr{T}$ denotes the support of $\mathscr{T}$. If $t \in \Gamma(X, \mathscr{T})$, then Supp t denotes the support of t. For $x \in X$, t_x denotes the germ of t at x. By the annihilator-ideal-sheaf $\mathscr{A}$ of $\mathscr{T}$ we mean the ideal-sheaf $\mathscr{A}$ defined by $\mathscr{A}_x=\{s \in \mathscr{H}_x \mid s\mathscr{T}_x=0\}$ for $x \in X$. If $\theta: (X, \mathscr{H}) \to (X', \mathscr{H}')$ is a holomorphic map (i.e. a morphism of ringed spaces) from $(X, \mathscr{H})$ to another complex space $(X', \mathscr{H}')$, then $R^0\theta(\mathscr{T})$ denotes the zeroth direct image of $\mathscr{T}$ under θ. If $f \in \Gamma(X, \mathscr{H})$ and $x \in X$, we say that f vanished at x if f_x is not a unit in $\mathscr{H}_x$.

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I. Absolute gap-sheaves.

DEFINITION 1. Suppose $\mathscr{S}$ is an analytic subsheaf of an analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer. The ρth *gap-sheaf of* $\mathscr{S}$ *in* $\mathscr{T}$, denoted by $\mathscr{S}_{[\rho]\mathscr{T}}$, is the analytic subsheaf of $\mathscr{T}$ defined as follows: For $x \in X$, $s \in (\mathscr{S}_{[\rho]\mathscr{T}})_x$ if and only if there exist an open neighborhood U of x in X, a subvariety A in U of dimension $\leqq \rho$, and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t_y \in \mathscr{S}_y$ for $y \in U - A$.

Denote the set $\{x \in X \mid \mathscr{S}_x \neq (\mathscr{S}_{[\rho]\mathscr{T}})_x\}$ by $E^\rho(\mathscr{S}, \mathscr{T})$.

REMARK. When $\mathscr{S}$ and $\mathscr{T}$ are both coherent, then $x \in E^\rho(\mathscr{S}, \mathscr{T})$ if and only if $\mathscr{S}_x$ as an $\mathscr{H}_x$-submodule of $\mathscr{T}_x$ has an associated prime ideal of dimension $\leqq \rho$ (Theorem 4, [12]). $E^\rho(\mathscr{S}, \mathscr{T}) = \varnothing$ means that for every $x \in X$ $\mathscr{S}_x$ as an $\mathscr{H}_x$-submodule of $\mathscr{T}_x$ has no associated prime ideal of dimension $\leqq \rho$.

DEFINITION 2. Suppose $\mathscr{S}$ is an analytic subsheaf of an analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$ and A is a subvariety of X. Then the *gap-sheaf of* $\mathscr{S}$ *in* $\mathscr{T}$ *with respect to* A, denoted by $\mathscr{S}[A]_{\mathscr{T}}$, is defined as follows: For $x \in X$, $s \in (\mathscr{S}[A]_{\mathscr{T}})_x$ if and only if there exist an open neighborhood U of x in X and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t_y \in \mathscr{S}_y$ for $y \in U - A$.

PROPOSITION 1. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer. Then $\mathscr{S}_{[\rho]\mathscr{T}}$ is coherent and $E^\rho(\mathscr{S}, \mathscr{T})$ is a subvariety of dimension $\leqq \rho$ in X.*

Proof. See Theorem 3 [12]. This can also be derived easily from Satz 3 [13]. Q.E.D.

PROPOSITION 2. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$ and A is a subvariety of X. Then $\mathscr{S}[A]_{\mathscr{T}}$ is coherent.*

Proof. See Theorem 1 [12]. This can also be derived easily from [13, Satz 9]. Q.E.D.

DEFINITION 3. Suppose $\mathscr{F}$ is an analytic sheaf on a complex space X and ρ is a nonnegative integer. The ρth *absolute gap-sheaf of* $\mathscr{F}$, denoted by $\mathscr{F}^{[\rho]}$, is the analytic sheaf on X defined by the following presheaf: Suppose $U \subset V$ are open subsets of X. Then

$$\mathscr{F}^{[\rho]}(U) = \operatorname*{ind\,lim}_{A \in \mathfrak{A}(U)} \Gamma(U - A, \mathscr{F}),$$

where $\mathfrak{A}(U)$ is the directed set of all analytic subvarieties in U of dimension $\leqq \rho$ directed under inclusion. $\mathscr{F}^{[\rho]}(V) \to \mathscr{F}^{[\rho]}(U)$ is induced by restriction.

REMARKS. (i) $\mathscr{F}^{[\rho]} = (\mathscr{F}/0_{[\rho]\mathscr{F}})^{[\rho]}$, where 0 is the zero-subsheaf of $\mathscr{F}$.

(ii) There is a natural sheaf-homomorphism $\mu: \mathscr{F} \to \mathscr{F}^{[\rho]}$. The kernel of μ is $0_{[\rho]\mathscr{F}}$. When $E^\rho(0, \mathscr{F}) = \varnothing$, μ is injective and we can regard $\mathscr{F}$ as a subsheaf of $\mathscr{F}^{[\rho]}$. In this case we denote the set $\{x \in X \mid \mathscr{F}_x \neq (\mathscr{F}^{[\rho]})_x\}$ by $E^\rho(\mathscr{F})$.

LEMMA 1. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a reduced complex space $(X, \mathcal{O})$ of pure dimension n. Suppose $0 \leqq \rho \leqq n-2$. If $E^{n-1}(0, \mathscr{F}) = \varnothing$, then $\mathscr{F}^{[\rho]}$ is coherent and $E^\rho(\mathscr{F})$ is a subvariety of dimension $\leqq \rho$.*

Proof. Let $\pi: (\tilde{X}, \tilde{\mathcal{O}}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\tilde{\mathscr{F}}$ be the inverse image of $\mathscr{F}$ under π (Definition 8, [6]). Let $\mathscr{T}$ be the torsion-subsheaf of $\tilde{\mathscr{F}}$ and $\mathscr{G} = \tilde{\mathscr{F}}/\mathscr{T}$. Let $Y = \operatorname{Supp} \mathscr{T}$. $\mathscr{T}$ and $\mathscr{G}$ are both coherent and $\mathscr{G}$ is torsion-free (Proposition 6, [1]). dim $Y \leqq n-1$ (Proposition 7, [1]). We claim that

(1) $\mathscr{G}^{[\rho]}$ is coherent and $E^\rho(\mathscr{G})$ is a subvariety of dimension $\leqq \rho$ in $\tilde{X}$.

Take $x \in \tilde{X}$. On some open neighborhood U of x in $\tilde{X}$ $\mathscr{G}$ can be regarded as a coherent subsheaf of $\tilde{\mathcal{O}}^p$ for some p (Proposition 9, [1]). It is clear that $\mathscr{G}^{[\rho]}$ is isomorphic to $\mathscr{G}_{[\rho]\tilde{\mathcal{O}}^p}$ on U and $E^\rho(\mathscr{G}, \tilde{\mathcal{O}}^p) \cap U = E^\rho(\mathscr{G}) \cap U$. (1) follows from Proposition 1.

Let $\mathscr{F}^* = R^0\pi(\tilde{\mathscr{F}})$, $\mathscr{G}^* = R^0\pi(\mathscr{G})$, and $(\mathscr{G}^{[\rho]})^* = R^0\pi(\mathscr{G}^{[\rho]})$. Let $\alpha: \mathscr{F}^* \to \mathscr{G}^*$ and $\beta: \mathscr{G}^* \to (\mathscr{G}^{[\rho]})^*$ be induced respectively by the quotient map $\tilde{\mathscr{F}} \to \mathscr{G}$ and the inclusion map $\mathscr{G} \to \mathscr{G}^{[\rho]}$. We have a natural sheaf-homomorphism $\lambda: \mathscr{F} \to \mathscr{F}^*$ (Satz 7(b), [6]). Let Z be the set of all singular points of X. Let $\mathscr{K}$ be the kernel of $\alpha\lambda$. Then $\operatorname{Supp} \mathscr{K} \subset Z \cup \pi(Y)$. Since $E^{n-1}(0, \mathscr{F}) = \varnothing$ and $\dim \operatorname{Supp} \mathscr{K} \leqq n-1$, $\mathscr{K} = 0$. $\gamma = \beta\alpha\lambda: \mathscr{F} \to (\mathscr{G}^{[\rho]})^*$ is injective. It is easily seen that $((\mathscr{G}^{[\rho]})^*)^{[\rho]} = (\mathscr{G}^{[\rho]})^*$. γ induces a sheaf-monomorphism $\gamma_1: \mathscr{F}^{[\rho]} \to (\mathscr{G}^{[\rho]})^*$. $\mathscr{F}^{[\rho]} \approx \gamma_1(\mathscr{F}^{[\rho]}) = \gamma(\mathscr{F})_{[\rho](\mathscr{G}^{[\rho]})^*}$ and $E^\rho(\mathscr{F}) = E^\rho(\gamma(\mathscr{F}), (\mathscr{G}^{[\rho]})^*)$. Since by Proposition 1 $\gamma(\mathscr{F})_{[\rho](\mathscr{G}^{[\rho]})^*}$ is coherent and $E^\rho(\gamma(\mathscr{F}), (\mathscr{G}^{[\rho]})^*)$ is a subvariety of dimension $\leqq \rho$ in X, the Lemma follows. Q.E.D.

LEMMA 2. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$. Suppose $x \in X$ and $f \in \mathscr{H}_x$ such that for every nonnegative integer ρ either $x \notin E^\rho(0, \mathscr{F})$ or f does not vanish identically on any branch-germ of $E^\rho(0, \mathscr{F})$ at x. Then f is not a zero-divisor for $\mathscr{F}_x$.*

Proof. Suppose the contrary. Then there exist $s \in \Gamma(U, \mathscr{F})$ and $g \in \Gamma(U, \mathscr{H})$ for some open neighborhood U of x such that $g_x = f$, $gs = 0$, and $s_x \neq 0$. Let $Z = \operatorname{Supp} s$ and $\dim Z_x = \rho$. By shrinking U, we can assume that $\dim Z = \rho$. Hence $Z \subset E^\rho(0, \mathscr{F})$. Since $\dim E^\rho(0, \mathscr{F}) \leqq \rho$, the union Z_0 of all ρ-dimensional branches of Z is equal to the union of some ρ-dimensional branches of $E^\rho(0, \mathscr{F}) \cap U$. By assumption g does not vanish identically on Z_0. For some $y \in Z_0$, g_y is a unit in $\mathscr{H}_y$. $s_y = 0$, contradicting that $Z = \operatorname{Supp} s$. Q.E.D.

LEMMA 3. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space X and ρ is a nonnegative integer. If $E^\rho(0, \mathscr{F}) = \varnothing$, then for any nonnegative integer σ either $E^\sigma(0, \mathscr{F}) = \varnothing$ or every branch of $E^\sigma(0, \mathscr{F})$ has dimension $> \rho$.*

Proof. Suppose Y is a nonempty m-dimensional branch of $E^\sigma(0, \mathscr{F})$ for some nonnegative integer σ such that $m \leqq \rho$. Take a Stein open subset U of X such that $U \cap E^\sigma(0, \mathscr{F}) = U \cap Y \neq \varnothing$. Take $x \in U \cap Y$. Since $(0_{[\sigma]\mathscr{F}})_x \neq 0$, there exists

$s \in \Gamma(U, 0_{[\sigma]\mathscr{F}})$ such that $s_x \neq 0$. Supp $s \subset E^\sigma(0, \mathscr{F}) \cap U = U \cap Y$. dim Supp $s \leqq \rho$. Hence $s \in \Gamma(U, 0_{[\rho]\mathscr{F}})$. $x \in E^\rho(0, \mathscr{F})$, contradicting that $E^\rho(0, \mathscr{F}) = \varnothing$. Q.E.D.

LEMMA 4. *Suppose $\mathscr{F}_i$, $1 \leqq i \leqq 3$, are coherent analytic sheaves on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer such that $E^\rho(0, \mathscr{F}_i) = 0$ for $1 \leqq i \leqq 3$. Suppose $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \xrightarrow{\eta} \mathscr{F}_3 \to 0$ is an exact sequence of sheaf-homomorphisms. If $(\mathscr{F}_i)^{[\rho]}$ is coherent and $E^\rho(\mathscr{F}_i)$ is a subvariety of dimension $\leqq \rho$ for $i = 1, 3$, then $(\mathscr{F}_2)^{[\rho]}$ is coherent and $E^\rho(\mathscr{F}_2)$ is a subvariety of dimension $\leqq \rho$.*

Proof. Let $X_i = E^\rho(\mathscr{F}_i)$, $i = 1, 3$. The problem is local in nature. Take $x_0 \in X$ and take an open Stein neighborhood U of x_0 in X. $\mathscr{F}_i$ is a coherent analytic subsheaf of $(\mathscr{F}_i)^{[\rho]}$, $i = 1, 3$. Let $\mathscr{A}_i = \mathscr{F}_i : (\mathscr{F}_i)^{[\rho]}$, $i = 1, 3$. $E(\mathscr{A}_i, \mathscr{H}) = X_i$, $i = 1, 3$. Let $\mathscr{I}_i$ be the ideal-sheaf for X_i, $i = 1, 3$. By Hilbert Nullstellensatz, after shrinking U, we can find a natural number m such that $\mathscr{I}_i^m \subset \mathscr{A}_i$ on U, $i = 1, 3$. By Lemma 3 for any nonnegative integer σ every nonempty branch of $E^\sigma(0, \mathscr{F}_2)$ has dimension $> \rho$. Since dim $X_i \leqq \rho$, $i = 1, 3$, we can choose $f \in \Gamma(U, \mathscr{I}_1^m \cap \mathscr{I}_2^m)$ such that f_{x_0} does not vanish identically on any nonempty branch-germ of $E^\sigma(0, \mathscr{F}_2)$ at x_0 for any nonnegative integer σ. By Lemma 2 f_{x_0} is not a zero-divisor for $(\mathscr{F}_2)_{x_0}$. Let $\mathscr{K}$ be the kernel of the sheaf-homomorphism $\alpha: \mathscr{F}_2 \to \mathscr{F}_2$ on U defined by multiplication by f. Then $\mathscr{K}_{x_0} = 0$. By shrinking U, we can assume that $\mathscr{K} = 0$ on U. α induces a sheaf-monomorphism $\beta: (\mathscr{F}_2)^{[\rho]} \to (\mathscr{F}_2)^{[\rho]}$. Let $\gamma = \beta \circ \beta$. We claim that $\gamma((\mathscr{F}_2)^{[\rho]}) \subset \mathscr{F}_2$ on U. Take $s \in ((\mathscr{F}_2)^{[\rho]})_x$ for some $x \in U$. s is defined by some $t \in \Gamma(W - A, \mathscr{F}_2)$, where W is an open neighborhood of x in U and A is a subvariety of dimension $\leqq \rho$ in W. $\eta(t) \in \Gamma(W - A, \mathscr{F}_3)$ defines an element a of $((\mathscr{F}_3)^{[\rho]})_x$. $f_x a \in (\mathscr{F}_3)_x$. By shrinking W we can find $u \in \Gamma(W, \mathscr{F}_3)$ such that u agrees with $f\eta(t)$ on $W - A$ and we can find $v \in \Gamma(W, \mathscr{F}_2)$ such that $\eta(v) = u$. $\eta(v - ft) = 0$ on $W - A$. $v - ft$ defines an element b of $((\mathscr{F}_1)^{[\rho]})_x$. $f_x b \in (\mathscr{F}_1)_x$. By shrinking W we can find $w \in \Gamma(W, \mathscr{F}_1)$ such that w agrees with $f(v - ft)$ on $W - A$. $f^2 t = fv - w$ on $W - A$. $\gamma(s) = \beta(v_x) - w_x \in (\mathscr{F}_2)_x$. Hence $\gamma((\mathscr{F}_2)^{[\rho]}) \subset \mathscr{F}_2$. It is easily seen that $\gamma((\mathscr{F}_2)^{[\rho]}) = \gamma(\mathscr{F}_2)_{[\rho]\mathscr{F}_2}$ on U and $E^\rho(\mathscr{F}_2) \cap U = E^\rho(\gamma(\mathscr{F}_2), \mathscr{F}_2) \cap U$. The Lemma follows from Proposition 1. Q.E.D.

LEMMA 5. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ of pure dimension n and $0 \leqq \rho \leqq n-2$. If $E^{n-1}(0, \mathscr{F}) = \varnothing$, then $\mathscr{F}^{[\rho]}$ is coherent and $E^\rho(\mathscr{F})$ is a subvariety of dimension $\leqq \rho$.*

Proof. Let $\mathscr{K}$ be the subsheaf of all nilpotent elements of $\mathscr{H}$ and $\mathscr{O} = \mathscr{H}/\mathscr{K}$. Since the lemma is local in nature, we can suppose that for some nonnegative integer k $\mathscr{K}^k = 0$. For $0 \leqq l \leqq k$ define $\mathscr{F}^{(l)}$ inductively as follows: $\mathscr{F}^{(0)} = \mathscr{F}$ and, for $1 \leqq l \leqq k$, $\mathscr{F}^{(l)} = (\mathscr{K}\mathscr{F}^{(l-1)})_{[n-1]\mathscr{F}^{(l-1)}}$. Let $Y = \bigcup_{l=1}^k E^{n-1}(\mathscr{K}\mathscr{F}^{(l-1)}, \mathscr{F}^{(l-1)})$. Y is a subvariety of dimension $\leqq n-1$. On $X - Y$ $\mathscr{F}^{(l)} = \mathscr{K}\mathscr{F}^{(l-1)}$ for $1 \leqq l \leqq k$. Hence $\mathscr{F}^{(k)} = 0$ on $X - Y$. Since $\mathscr{F}^{(k)} \subset \mathscr{F}$ and $E^{n-1}(0, \mathscr{F}) = \varnothing$, $\mathscr{F}^{(k)} = 0$. From the definition of $\mathscr{F}^{(l)}$ we see that $E^{n-1}(\mathscr{F}^{(l)}, \mathscr{F}^{(l-1)}) = \varnothing$ for $1 \leqq l \leqq k$. Hence $E^{n-1}(0, \mathscr{F}^{(l-1)}/\mathscr{F}^{(l)}) = \varnothing$ for $1 \leqq l \leqq k$. $E^{n-1}(0, \mathscr{F}) = \varnothing$ implies that $E^{n-1}(0, \mathscr{F}^{(l)}) = \varnothing$, $0 \leqq l \leqq k$. Since $\mathscr{K}\mathscr{F}^{(l-1)} \subset \mathscr{F}^{(l)}$, $\mathscr{F}^{(l-1)}/\mathscr{F}^{(l)}$ can be regarded as a coherent analytic sheaf on $(X, \mathscr{O})$,

$1 \leqq l \leqq k$. By Lemma 1 $(\mathscr{F}^{(l-1)}/\mathscr{F}^{(l)})^{[\rho]}$ is coherent and $E^\rho(\mathscr{F}^{(\rho-1)}/\mathscr{F}^{(\rho)})$ is a subvariety of dimension $\leqq \rho$. Since $\mathscr{F}^{(k)}=0$, from Lemma 4 and the exact sequences $0 \to \mathscr{F}^{(l)} \to \mathscr{F}^{(l-1)} \to \mathscr{F}^{(l-1)}/\mathscr{F}^{(l)} \to 0$, $1 \leqq l \leqq k$, we conclude by backward induction on l that $(\mathscr{F}^{(l)})^{[\rho]}$ is coherent and $E^\rho(\mathscr{F}^{(l)})$ is a subvariety of dimension $\leqq \rho$ for $0 \leqq l \leqq k$. The Lemma follows from $\mathscr{F}=\mathscr{F}^{(0)}$. Q.E.D.

LEMMA 6. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer. Let Y be the union of $(\rho+1)$-dimensional branches of $E^{\rho+1}(0, \mathscr{F})$. Then for $x \in Y$ $(\mathscr{F}^{[\rho]})_x$ is not finitely generated over $\mathscr{H}_x$.*

Proof. We can assume that $Y \neq \varnothing$. Let $\mathscr{G}=\mathscr{F}/0_{[\rho]\mathscr{F}}$. Since $E^\rho(0, \mathscr{G})=\varnothing$, by Lemma 3 and Proposition 1 every branch of $E^{\rho+1}(0, \mathscr{G})$ is $(\rho+1)$-dimensional. Since $\mathscr{G}$ agrees with $\mathscr{F}$ on $X-E^\rho(0, \mathscr{F})$, $E^{\rho+1}(0, \mathscr{G})-E^\rho(0, \mathscr{F})=E^{\rho+1}(0, \mathscr{F})-E^\rho(0, \mathscr{F})$. $\dim E^\rho(0, \mathscr{F}) \leqq \rho$ implies that $E^{\rho+1}(0, \mathscr{G})=Y$.

Fix $x \in Y$. Suppose $(\mathscr{F}^{[\rho]})_x$ is finitely generated over $\mathscr{H}_x$. Let $\mathscr{S}=0_{[\rho+1]\mathscr{G}}$. Since $E^\rho(0, \mathscr{S}) \subset E^\rho(0, \mathscr{G})=\varnothing$, $\mathscr{S} \subset \mathscr{S}^{[\rho]} \subset \mathscr{G}^{[\rho]}=\mathscr{F}^{[\rho]}$. Since $\operatorname{Supp} \mathscr{S}=E^{\rho+1}(0, \mathscr{G})=Y$, $(\mathscr{S}^{[\rho]})_x$ is a nonzero finitely generated $\mathscr{H}_x$-module. Let $(\mathscr{S}^{[\rho]})_x$ be generated by $s_1, \ldots, s_m \in (\mathscr{S}^{[\rho]})_x$. For some open neighborhood U of x in X and for some subvariety A of dimension $\leqq \rho$ in U s_i is induced by $t_i \in \Gamma(U-A, \mathscr{S})$, $1 \leqq i \leqq m$. By shrinking U, we can choose $f \in \Gamma(U, \mathscr{H})$ such that $W=Z(f) \cap Y$ is a subvariety of dimension ρ in U and $x \in Z(f)$, where $Z(f)=\{y \in U \mid f_y \text{ is not a unit in } \mathscr{H}_y\}$. There exists a unique $g \in \Gamma(U-Z(f), \mathscr{H})$ such that $gf=1$ on $U-Z(f)$. For $1 \leqq i \leqq m$ define $u_i \in \Gamma(U-(A \cup W), \mathscr{S})$ by $(u_i)_y=0$ for $y \in U-Y$ and $(u_i)_y=(gt_i)_y$ for $y \in Y \cap (U-(A \cup W))$. u_i induces $v_i \in (\mathscr{S}^{[\rho]})_x$, $1 \leqq i \leqq m$. $f_x v_i=s_i$, $1 \leqq i \leqq m$. For some $\alpha_{ij} \in \mathscr{H}_x$, $v_i=\sum_{j=1}^m \alpha_{ij} s_j$, $1 \leqq i \leqq m$. $s_i=f_x v_i=\sum_{j=1}^m \alpha_{ij} f_x s_j$, $1 \leqq i \leqq m$. $(\mathscr{S}^{[\rho]})_x=f_x(\mathscr{S}^{[\rho]})_x$. Since f_x is not a unit in $\mathscr{H}_x$, by [8, (4.1)] we have $(\mathscr{S}^{[\rho]})_x=0$ (contradiction). Q.E.D.

THEOREM 1. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer. Then $\mathscr{F}^{[\rho]}$ is coherent if and only if* $\dim E^{\rho+1}(0, \mathscr{F}) < \rho+1$. *In that case $E^\rho(\mathscr{F}/0_{[\rho]\mathscr{F}})$ is a subvariety of dimension $\leqq \rho$.*

Proof. It follows from Lemma 6 that, if $\mathscr{F}^{[\rho]}$ is coherent, then $\dim E^{\rho+1}(0, \mathscr{F}) < \rho+1$.

Suppose now $\dim E^{\rho+1}(0, \mathscr{F}) < \rho+1$. We are going to prove that $\mathscr{F}^{[\rho]}$ is coherent and $E^\rho(\mathscr{F}/0_{[\rho]\mathscr{F}})$ is a subvariety of dimension $\leqq \rho$ in X. Since $\mathscr{F}$ agrees with $\mathscr{F}/0_{[\rho]\mathscr{F}}$ on $X-E^\rho(0, \mathscr{F})$, $E^{\rho+1}(0, \mathscr{F}/0_{[\rho]\mathscr{F}})$ is contained in the subvariety $E^\rho(0, \mathscr{F}) \cup E^{\rho+1}(0, \mathscr{F})$ of dimension $\leqq \rho$. $E^\rho(0, \mathscr{F}/0_{[\rho]\mathscr{F}})=\varnothing$ implies $E^{\rho+1}(0, \mathscr{F}/0_{[\rho]\mathscr{F}})=\varnothing$ by Lemma 3. Since $\mathscr{F}^{[\rho]}=(\mathscr{F}/0_{[\rho]\mathscr{F}})^{[\rho]}$, by replacing $\mathscr{F}$ by $\mathscr{F}/0_{[\rho]\mathscr{F}}$, we can assume that $E^{\rho+1}(0, \mathscr{F})=\varnothing$. Since the problem is local in nature, we can suppose that X is of finite dimension n. If $n<\rho+2$, $E^{\rho+1}(0, \mathscr{F})=\varnothing$ implies that $\mathscr{F}=0$. $\mathscr{F}^{[\rho]}=0$ is coherent and $E^\rho(\mathscr{F})=\varnothing$. So we can assume that $n \geqq \rho+2$. For $\rho+1 \leqq m \leqq n$ let $\mathscr{G}^{(m)}=0_{[m]\mathscr{F}}$. $\mathscr{G}^{(\rho+1)}=0$, because $E^{\rho+1}(0, \mathscr{F})=\varnothing$. For $\rho+2 \leqq m \leqq n$ let $X_m=\operatorname{Supp} \mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}$. X_m is the union of all m-dimensional branches of $E^m(0, \mathscr{F})$,

$\rho+2\leqq m\leqq n$. $E^{m-1}(0, \mathscr{G}^{(m)}/\mathscr{G}^{(m-1)})=\varnothing$ for $\rho+2\leqq m\leqq n$. For $\rho+2\leqq m\leqq n$ let $\mathscr{A}^{(m)}$ be the annihilator-ideal-sheaf for $\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}$. Then $(\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}) \mid X_m$ can be regarded as a coherent analytic sheaf on the complex space $(X_m, (\mathscr{H}/\mathscr{A}^{(m)}) \mid X_m)$ which is either empty or of pure dimension m, $\rho+2\leqq m\leqq n$. By Lemma 5

$$(\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)})^{[\rho]} \approx ((\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}) \mid X_m)^{[\rho]}$$

is coherent and $E^{\rho}(\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)})=E^{\rho}((\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}) \mid X_m)$ is a subvariety of dimension $\leqq\rho$, $\rho+2\leqq m\leqq n$. Since $\mathscr{G}^{(\rho+2)}=\mathscr{G}^{(\rho+2)}/\mathscr{G}^{(\rho+1)}$, from Lemma 4 and the exact sequences $0\to\mathscr{G}^{(m-1)}\to\mathscr{G}^{(m)}\to\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}\to 0$, $\rho+3\leqq m\leqq n$, we conclude by induction on m that $(\mathscr{G}^{(m)})^{[\rho]}$ is coherent and $E^{\rho}(\mathscr{G}^{(m)})$ is a subvariety of dimension $\leqq\rho$, $\rho+2\leqq m\leqq n$. The Theorem follows from $\mathscr{F}=\mathscr{G}^{(n)}$. Q.E.D.

Corollary 1. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space X, ρ is a nonnegative integer, and $x\in X$. $\mathscr{F}^{[\rho]}$ is coherent at x if and only if x does not belong to a $(\rho+1)$-dimensional branch of $E^{\rho+1}(0,\mathscr{F})$. Hence the set of points where $\mathscr{F}^{[\rho]}$ is not coherent is either empty or it is a subvariety of pure dimension $\rho+1$.*

Remark. Under the assumption of Corollary 1 to Theorem 2 x does not belong to a $(\rho+1)$-dimensional branch of $E^{\rho+1}(0,\mathscr{F})$ if and only if the zero submodule of $\mathscr{F}_x$ has no associated prime ideal of dimension $\rho+1$ [12, Theorem 4]. This gives us an algebraic criterion for the coherence of $\mathscr{F}^{[\rho]}$ at x.

Corollary 2. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space X and ρ is a nonnegative integer. Let $\mu\colon\mathscr{F}\to\mathscr{F}^{[\rho]}$ be the natural sheaf-homomorphism. Then $Z=\{x\in X \mid \mu_x$ is not surjective$\}$ is a subvariety of dimension $\leqq\rho+1$.*

Proof. Let Y be the union of all $(\rho+1)$-dimensional branches of $E^{\rho+1}(0,\mathscr{F})$. By Lemma 6 $Y\subset Z$. Since $\mathscr{F}^{[\rho]}$ agrees with $(\mathscr{F}/0_{[\rho+1]\mathscr{F}})^{[\rho]}$ on $X-Y$, $Z-Y=E^{\rho}(\mathscr{F}/0_{[\rho+1]\mathscr{F}})-Y$. $Z=Y\cup E^{\rho}(\mathscr{F}/0_{[\rho+1]\mathscr{F}})$ is a subvariety of dimension $\leqq\rho+1$. Q.E.D.

Remark. Corollary 2 to Theorem 1 can be stated alternatively in the following way: *The set of points where we cannot always remove closed singularities contained in subvarieties of dimension ρ for local sections of a coherent analytic sheaf $\mathscr{F}$ satisfying $E^{\rho}(0,\mathscr{F})=\varnothing$ is a subvariety of dimension $\leqq\rho+1$.*

The weaker statement that this set of points is contained in a subvariety of dimension $\leqq\rho+1$ is an easy consequence of Satz III, [9] and Satz 5, [10].

II. Extension of coherent sheaves.

Suppose S is a subvariety of a complex space X and $\mathscr{F}$ is a coherent analytic sheaf on $X-S$. *$\mathscr{F}$ is said to satisfy $(*)_{X,S}$ if for every $x\in S$ there exists some open neighborhood U of x in X such that $\Gamma(U-S,\mathscr{F})$ generates $\mathscr{F}$ on $U-S$.*

Lemma 7. *Suppose S is a subvariety of codimension $\geqq 2$ in a reduced complex space $(X,\mathscr{O})$ of pure dimension n. Let $\theta\colon X-S\to X$ be the inclusion map. Suppose $\mathscr{F}$*

is a coherent analytic sheaf on $X-S$ such that $E^{n-1}(0, \mathscr{F})=\varnothing$. If $\mathscr{F}$ satisfies $(*)_{X,S}$, then $R^0\theta(\mathscr{F})$ is coherent.

Proof. Let $\pi: (\tilde{X}, \tilde{\mathcal{O}}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\tilde{S}=\pi^{-1}(S)$ and $\pi'=\pi|(\tilde{X}-\tilde{S})$. Let $\tilde{\theta}: \tilde{X}-\tilde{S} \to \tilde{X}$ be the inclusion map. Let $\tilde{\mathscr{F}}$ be the inverse image of $\mathscr{F}$ under π'. Let $\mathscr{T}$ be the torsion-subsheaf of $\tilde{\mathscr{F}}$, $\mathscr{G}=\tilde{\mathscr{F}}/\mathscr{T}$, and $Y=\operatorname{Supp}\mathscr{T}$. Since $\mathscr{F}$ satisfies $(*)_{X,S}$, $\tilde{\mathscr{F}}$ satisfies $(*)_{\tilde{X},\tilde{S}}$. This implies that $\mathscr{G}$ satisfies $(*)_{\tilde{X},\tilde{S}}$. By Theorem 1, [11] $R^0\tilde{\theta}(\mathscr{G})$ is coherent on $\tilde{X}$. Let $\mathscr{F}^*=R^0\pi'(\tilde{\mathscr{F}})$ and $\mathscr{G}^*=R^0\pi(R^0\tilde{\theta}(\mathscr{G}))$. $\mathscr{G}^*$ is coherent on X. Let the sheaf-homomorphism $\alpha: \mathscr{F}^* \to \mathscr{G}^*$ on $X-S$ be induced by the quotient map $\tilde{\mathscr{F}} \to \mathscr{G}$. We have a natural sheaf-homomorphism $\lambda: \mathscr{F} \to \mathscr{F}^*$. Let Z be the set of all singular points on X. Let $\mathscr{K}$ be the kernel of $\alpha\lambda$. Then $\operatorname{Supp}\mathscr{K} \subset Z \cup \pi(Y)$. Since $E^{n-1}(0, \mathscr{F})=\varnothing$ and $\dim\operatorname{Supp}\mathscr{K} \leqq n-1$, $\mathscr{K}=0$. $\alpha\lambda$ is injective. Since $R^0\theta(\mathscr{G}^* \mid X-S)=\mathscr{G}^*$, $\alpha\lambda$ induces a sheaf-monomorphism $\beta: R^0\theta(\mathscr{F}) \to \mathscr{G}^*$. Take $x \in S$. There exists an open neighborhood U of x in X such that $\Gamma(U-S, \mathscr{F})$ generates $\mathscr{F}$ on $U-S$. For $s \in \Gamma(U-S, \mathscr{F})$ let $\hat{s} \in \Gamma(U, \mathscr{G}^*)$ be the unique extension of $\alpha\lambda(s)$. $\{\hat{s} \mid s \in \Gamma(U-S, \mathscr{F})\}$ generates a coherent analytic subsheaf $\mathscr{S}$ of $\mathscr{G}^*$ on U. On U $\beta(R^0\theta(\mathscr{F}))=\mathscr{S}[S]_{\mathscr{G}^*}$. By Proposition 2 $\mathscr{S}[S]_{\mathscr{G}^*}$ is coherent. Hence $R^0\theta(\mathscr{F})$ is coherent. Q.E.D.

LEMMA 8. *Suppose S is a subvariety in a complex space $(X, \mathscr{H})$. Let $\theta: X-S \to X$ be the inclusion map. Suppose $\mathscr{F}_i$, $1 \leqq i \leqq 3$, are coherent analytic sheaves on $X-S$ such that $R^0\theta(\mathscr{F}_3)$ is coherent. Suppose $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \xrightarrow{\eta} \mathscr{F}_3 \to 0$ is an exact sequence of sheaf-homomorphisms on $X-S$. If $\mathscr{F}_2$ satisfies $(*)_{X,S}$, then $\mathscr{F}_1$ satisfies $(*)_{X,S}$.*

Proof. Take $x \in S$. There is an open neighborhood U of x in X such that $\Gamma(U-S, \mathscr{F}_2)$ generates $\mathscr{F}_2$ on $U-S$. Let W be a Stein open neighborhood of x in U. We claim that $\Gamma(W-S, \mathscr{F}_1)$ generates $\mathscr{F}_1$ on $W-S$. Take $y \in W-S$. There exist $s_i \in \Gamma(U-S, \mathscr{F}_2)$, $1 \leqq i \leqq m$, generating $(\mathscr{F}_2)_y$. Define a sheaf-homomorphism $\varphi: \mathscr{H}^m \to \mathscr{F}_2$ on $U-S$ by $\varphi(\alpha_1, \dots, \alpha_m)=\sum_{i=1}^m \alpha_i(s_i)_z$ for $\alpha_1, \dots, \alpha_m \in \mathscr{H}_z$ and $z \in U-S$. $\eta(s_i)$ can be extended uniquely to an element of $\Gamma(U, R^0\theta(\mathscr{F}_3))$, $1 \leqq i \leqq m$. There is a unique sheaf-homomorphism $\psi: \mathscr{H}^m \to R^0\theta(\mathscr{F}_3)$ on U which agrees with $\eta\varphi$ on $U-S$. Let $\mathscr{K}$ be the kernel of ψ. $\mathscr{K}$ is coherent. There exist $u_i \in \Gamma(W, \mathscr{K})$, $1 \leqq i \leqq n$, generating $\mathscr{K}_y$. Let $v_i=\varphi(u_i \mid (W-S))$, $1 \leqq i \leqq n$. Then $v_i \in \Gamma(W-S, \mathscr{F}_3)$, $1 \leqq i \leqq n$, and $(\mathscr{F}_3)_y$ is generated by $v_1, \dots, v_n$. Q.E.D.

LEMMA 9. *Suppose S is a subvariety of dimension ρ in a complex space X. Let $\theta: X-S \to X$ be the inclusion map. Suppose $\mathscr{F}_i$, $1 \leqq i \leqq 3$, are coherent analytic sheaves on $X-S$ such that $R^0\theta(\mathscr{F}_j)$ is coherent for $j=1, 3$. Suppose $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \xrightarrow{\eta} \mathscr{F}_3 \to 0$ is an exact sequence of sheaf-homomorphisms on $X-S$. If $\mathscr{F}_2$ satisfies $(*)_{X,S}$ and $E^{\rho+1}(0, \mathscr{F}_2)=\varnothing$, then $R^0\theta(\mathscr{F}_2)$ is coherent.*

Proof. Take $x \in S$. We need only prove that $R^0\theta(\mathscr{F}_2)$ is coherent at x. There is a Stein open neighborhood U of x in X such that $\Gamma(U-S, \mathscr{F}_2)$ generates $\mathscr{F}_2$ on $U-S$.

The exact sequence $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \xrightarrow{\eta} \mathscr{F}_3 \to 0$ induces the exact sequence $0 \to R^0\theta(\mathscr{F}_1) \to R^0\theta(\mathscr{F}_2) \xrightarrow{\eta'} R^0\theta(\mathscr{F}_3)$. For $s \in \Gamma(U-S, \mathscr{F}_2)$ let $\tilde{s} \in \Gamma(U, R^0\theta(\mathscr{F}_2))$ be the unique extension of s and let $\hat{s}=\eta'(\tilde{s})$. Let $\mathscr{S}$ be the subsheaf of $R^0\theta(\mathscr{F}_2)$ on U generated by $\{\tilde{s} \mid s \in \Gamma(U-S, \mathscr{F}_2)\}$ and $\mathscr{T}$ be the subsheaf of $R^0\theta(\mathscr{F}_3)$ on U generated by

$$\{\hat{s} \mid s \in \Gamma(U-S, \mathscr{F}_2)\}.$$

$\eta'(\mathscr{S})=\mathscr{T}$. Since $R^0\theta(\mathscr{F}_3)$ is coherent, $\mathscr{T}$ being generated by global sections is coherent. Since $R^0\theta(\mathscr{F}_1)$ is coherent and U is Stein, on U $R^0\theta(\mathscr{F}_1)$ is generated by $\Gamma(U, R^0\theta(\mathscr{F}_1)) \approx \Gamma(U-S, \mathscr{F}_1) \subset \Gamma(U-S, \mathscr{F}_2)$. $R^0\theta(\mathscr{F}_1) \subset \mathscr{S}$. We have an exact sequence $0 \to R^0\theta(\mathscr{F}_1) \xrightarrow{\xi} \mathscr{S} \xrightarrow{\eta''} \mathscr{T} \to 0$, where η'' is induced by η' and ξ is the inclusion map. Since $R^0\theta(\mathscr{F}_1)$ and $\mathscr{T}$ are both coherent, $\mathscr{S}$ is coherent. $E^{\rho+1}(0, \mathscr{S}) \subset E^{\rho+1}(0, \mathscr{F}_2) = \varnothing$. By Theorem 1 $\mathscr{S}^{[\rho]}$ is coherent. Since $\dim S=\rho$, $R^0\theta(\mathscr{S}^{[\rho]}) = \mathscr{S}^{[\rho]}$. The inclusion map $\mathscr{F}_2 \to \mathscr{S}$ on $U-S$ induces on U a sheaf-monomorphism $\beta\colon R^0\theta(\mathscr{F}_2) \to \mathscr{S}^{[\rho]}$. $\beta(R^0\theta(\mathscr{F}_2)) = \mathscr{S}[S]_{\mathscr{S}^{[\rho]}}$. Since $\mathscr{S}[S]_{\mathscr{S}^{[\rho]}}$ is coherent by Proposition 2, $R^0\theta(\mathscr{F}_2)$ is coherent on U. Q.E.D.

LEMMA 10. *Suppose S is a subvariety of codimension $\geqq 2$ in a complex space $(X, \mathscr{H})$ of pure dimension n. Let $\theta\colon X-S \to X$ be the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on $X-S$. If $\mathscr{F}$ satisfies $(*)_{X,S}$ and $E^{n-1}(0, \mathscr{F})=\varnothing$, then $R^0\theta(\mathscr{F})$ is coherent on X.*

Proof. Let $\mathscr{K}$ be the subsheaf of all nilpotent elements of $\mathscr{H}$ and $\mathscr{O}=\mathscr{H}/\mathscr{K}$. Since the Lemma is local in nature, we can suppose that for some nonnegative integer k $\mathscr{K}^k=0$. For $0 \leqq l \leqq k$ define coherent analytic sheaves $\mathscr{F}^{(l)}$ on $X-S$ inductively as follows: $\mathscr{F}^{(0)}=\mathscr{F}$ and, for $1 \leqq l \leqq k$, $\mathscr{F}^{(l)}=(\mathscr{K}\mathscr{F}^{(l-1)})_{[n-1]\mathscr{F}^{(l-1)}}$. Let

$$Y = \bigcup_{l=1}^{k} E^{n-1}(\mathscr{K}\mathscr{F}^{(l-1)}, \mathscr{F}^{(l-1)}).$$

Y is a subvariety in $X-S$ of dimension $\leqq n-1$. On $X-(S \cup Y)$, $\mathscr{F}^{(l)}=\mathscr{K}\mathscr{F}^{(l-1)}$ for $1 \leqq l \leqq k$. Hence $\mathscr{F}^{(k)}=0$ on $X-(S \cup Y)$. Since $\mathscr{F}^{(k)} \subset \mathscr{F}$ and $E^{n-1}(0, \mathscr{F})=\varnothing$, $\mathscr{F}^{(k)}=0$ on $X-S$. From the definition of $\mathscr{F}^{(l)}$ we see that $E^{n-1}(\mathscr{F}^{(l)}, \mathscr{F}^{(l-1)})=\varnothing$ for $1 \leqq i \leqq k$. Hence $E^{n-1}(0, \mathscr{F}^{(l-1)}/\mathscr{F}^{(l)})=0$ for $1 \leqq l \leqq k$. $E^{n-1}(0, \mathscr{F})=\varnothing$ implies that $E^{n-1}(0, \mathscr{F}^{(l)})=\varnothing$ for $1 \leqq l \leqq k$. Since $\mathscr{H}\mathscr{F}^{(l-1)} \subset \mathscr{F}^{(l)}$, $\mathscr{F}^{(l-1)}/\mathscr{F}^{(l)}$ can be regarded as a coherent analytic sheaf on $(X-S, \mathscr{O} \mid (X-S))$, $1 \leqq l \leqq k$.

Set $\mathscr{F}^{(k+1)}=0$. We are going to prove $(2)_l$ for $0 \leqq l \leqq k$ by induction on l:

$(2)_l$ $\qquad \mathscr{F}^{(l)}$ satisfies $(*)_{X,S}$ and $R^0\theta(\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)})$ is coherent.

Since $\mathscr{F}^{(0)}=\mathscr{F}$, $\mathscr{F}^{(0)}$ satisfies $(*)_{X,S}$. $\mathscr{F}^{(0)}/\mathscr{F}^{(1)}$ satisfies $(*)_{X,S}$. By Lemma 7

$$R^0\theta(\mathscr{F}^{(0)}/\mathscr{F}^{(1)})$$

is coherent. $(2)_0$ is true. Suppose for some $0 \leqq m < k$ $(2)_m$ is true. By Lemma 8 and the exact sequence $0 \to \mathscr{F}^{(m+1)} \to \mathscr{F}^{(m)} \to \mathscr{F}^{(m)}/\mathscr{F}^{(m+1)} \to 0$, we conclude that $\mathscr{F}^{(m+1)}$ satisfies $(*)_{X,S}$. Hence $\mathscr{F}^{(m+1)}/\mathscr{F}^{(m+2)}$ satisfies $(*)_{X,S}$. By Lemma 7

$$R^0\theta(\mathscr{F}^{(m+1)}/\mathscr{F}^{(m+2)})$$

is coherent. $(2)_{m+1}$ is true. Hence $(2)_l$ holds for $0 \leq l \leq k$.

Now we are going to prove $(3)_l$ for $0 \leq l \leq k$ by backward induction on l:

$(3)_l$ $\qquad R^0\theta(\mathscr{F}^{(l)})$ is coherent.

Since $\mathscr{F}^{(k)}=0$, $(3)_k$ is true. Suppose $(3)_m$ is true for some $0<m\leq k$. From $(2)_{m-1}$, $(3)_m$, Lemma 10 and the exact sequence $0 \to \mathscr{F}^{(m)} \to \mathscr{F}^{(m-1)} \to \mathscr{F}^{(m-1)}/\mathscr{F}^{(m)} \to 0$, we conclude that $(3)_{m-1}$ is true. Hence $(3)_l$ holds for $0\leq l\leq k$. The Lemma follows from $(3)_0$. Q.E.D.

Lemma 11. *Suppose S is a subvariety of dimension ρ in a complex space $(X, \mathscr{H})$. Suppose $\mathscr{F}$ is a coherent analytic sheaf on $X-S$ such that* Supp $\mathscr{F}$ *is a subvariety of pure dimension $n>\rho$ and $E^{n-1}(0, \mathscr{F})=\varnothing$. Then there exists a complex subspace $(Y, \mathscr{K})$ of pure dimension n in $(X, \mathscr{H})$ such that $Y-S=$* Supp $\mathscr{F}$ *and $\mathscr{F}|(Y-S)$ can be regarded as a coherent analytic sheaf on $(Y-S, \mathscr{K}|(Y-S))$.*

Proof. By [7, V.D.5] the topological closure Y of Supp $\mathscr{F}$ in X is a subvariety of pure dimension n. Let $Y=\bigcup_{\alpha\in A} Y_\alpha$ be the decomposition into irreducible branches. Let $\mathscr{I}_\alpha$ be the ideal-sheaf for Y_α, $\alpha\in A$. Choose $x_\alpha \in Y_\alpha-(S\cup(\bigcup_{\beta\in A,\beta\neq\alpha} Y_\beta))$. Let $\mathscr{A}$ be the annihilator-ideal-sheaf for $\mathscr{F}$. Then $E(\mathscr{A}, \mathscr{H}|(X-S))=Y-S$. By Hilbert Nullstellensatz, there exists a natural number m_α such that $(\mathscr{I}_\alpha^{m_\alpha})_{x_\alpha}\subset \mathscr{A}_{x_\alpha}$, $\alpha\in A$. Let $\mathscr{I}=\prod_{\alpha\in A}\mathscr{I}^{m_\alpha}$. Then $\mathscr{I}$ is coherent and $(\mathscr{I}\mathscr{F})_{x_\alpha}=0$ for $\alpha\in A$. Supp $\mathscr{I}\mathscr{F}$ is a subvariety of dimension $<n$ in $X-S$. $E^{n-1}(0,\mathscr{F})=\varnothing$ implies that $\mathscr{I}\mathscr{F}=0$. Set $\mathscr{K}=(\mathscr{H}/\mathscr{I})|Y$. Then $(Y, \mathscr{K})$ satisfies the requirements. Q.E.D.

Theorem 2. *Suppose S is a subvariety of dimension ρ in a complex space $(X, \mathscr{H})$. Let θ: $X-S\to X$ be the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on $X-S$ such that $E^{\rho+1}(0,\mathscr{F})=\varnothing$ or equivalently for every $x\in X-S$ the zero $\mathscr{H}_x$-submodule of $\mathscr{F}_x$ has no associated prime ideal of dimension $\leq\rho+1$. Then the following conditions are equivalent*:

(i) *$R^0\theta(\mathscr{F})$ is coherent.*

(ii) *There exists a coherent analytic sheaf on X which extends $\mathscr{F}$.*

(iii) *$\mathscr{F}$ satisfies $(*)_{X,S}$.*

Proof. It is clear that (i) implies (ii) and (ii) implies (iii). We need only prove that (iii) implies (i). Suppose $\mathscr{F}$ satisfies $(*)_{X,S}$. We are going to prove that $R^0\theta(\mathscr{F})$ is coherent. Since the problem is local in nature, we can suppose that X is of finite dimension n. If $n<\rho+2$, then $E^{\rho+1}(0,\mathscr{F})=\varnothing$ implies that $\mathscr{F}=0$. $R^0\theta(\mathscr{F})=0$ is coherent. So we can assume that $n\geq\rho+2$. For $\rho+1\leq m\leq n$ let $\mathscr{G}^{(m)}=0_{[m]\mathscr{F}}$. $\mathscr{G}^{(\rho+1)}=0$, because $E^{\rho+1}(0,\mathscr{F})=\varnothing$. For $\rho+2\leq m\leq n$ let $X_m=$ Supp $\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}$. Then X_m is the union of all m-dimensional branches of $E^m(0,\mathscr{F})$, $\rho+2\leq m\leq n$.

$E^{m-1}(0, \mathscr{G}^{(m)}/\mathscr{G}^{(m-1)}) = \varnothing$ for $\rho+2 \leqq m \leqq n$. By Lemma 11 there exists a complex subspace $(Y_m, \mathscr{H}_m)$ of pure dimension m in $(X, \mathscr{H})$ such that $Y_m - S = X_m$ and $(\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)})|(Y_m - S)$ can be regarded as a coherent analytic sheaf on

$$(Y_m - S, \mathscr{H}_m|(Y_m - S)), \rho+2 \leqq m \leqq n.$$

Let $\theta_m: Y_m - S \to Y_m$ be the inclusion map $\rho+2 \leqq m \leqq n$. $E^{\rho+1}(0, \mathscr{F}) = \varnothing$ implies that $E^{\rho+1}(0, \mathscr{G}^{(m)}) = 0$ for $\rho+2 \leqq m \leqq n$.

We are going to prove $(4)_m$ for $\rho+2 \leqq m \leqq n$ by backward induction on m:

$(4)_m$ $\mathscr{G}^{(m)}$ satisfies $(*)_{X,S}$ and $R^0\theta(\mathscr{G}^{(m)}/\mathscr{G}^{(m-1)})$ is coherent.

Since $\mathscr{G}^{(n)} = \mathscr{F}$, $\mathscr{G}^{(n)}$ satisfies $(*)_{X,S}$. $(\mathscr{G}^{(n)}/\mathscr{G}^{(n-1)})|(Y_n - S)$ satisfies $(*)_{Y_n, Y_n \cap S}$. By Lemma 10 $R^0\theta(\mathscr{G}^{(n)}/\mathscr{G}^{(n-1)}) \approx R^0\theta_n((\mathscr{G}^{(n)}/\mathscr{G}^{(n-1)})|(Y_n - S))$ is coherent. $(4)_n$ is true. Suppose for some $\rho+2 < q \leqq n$, $(4)_q$ is true. From Lemma 8, $(4)_q$, and the exact sequence $0 \to \mathscr{G}^{(q-1)} \to \mathscr{G}^{(q)} \to \mathscr{G}^{(q)}/\mathscr{G}^{(q-1)} \to 0$ we conclude that $\mathscr{G}^{(q-1)}$ satisfies $(*)_{X,S}$. $(\mathscr{G}^{(q-1)}/\mathscr{G}^{(q-2)})|(Y_{q-1} - S)$ satisfies $(*)_{Y_{q-1}, Y_{q-1} \cap S}$. By Lemma 10 $R^0\theta(\mathscr{G}^{(q-1)}/\mathscr{G}^{(q-2)}) \approx R^0\theta_{q-1}((\mathscr{G}^{(q-1)}/\mathscr{G}^{(q-2)})|(Y_{q-1} - S))$ is coherent. $(4)_{q-1}$ is true. Hence $(4)_m$ holds for $\rho+2 \leqq m \leqq n$.

Now we are going to prove $(5)_m$ for $\rho+1 \leqq m \leqq n$ by induction on m:

$(5)_m$ $R^0\theta(\mathscr{G}^{(m)})$ is coherent.

Since $\mathscr{G}^{(\rho+1)} = 0$, $(5)_{\rho+1}$ is true. Suppose $(5)_q$ is true for some $\rho+1 \leqq q < n$. From $(4)_{q+1}$, $(5)_q$, Lemma 9, and the exact sequence $0 \to \mathscr{G}^{(q)} \to \mathscr{G}^{(q+1)} \to \mathscr{G}^{(q+1)}/\mathscr{G}^{(q)} \to 0$ we conclude that $R^0\theta(\mathscr{G}^{(q+1)})$ is coherent. $(5)_{q+1}$ is true. Hence $(5)_m$ holds for $\rho+1 \leqq m \leqq n$. Since $\mathscr{G}^{(n)} = \mathscr{F}$, $(5)_n$ implies that $R^0\theta(\mathscr{F})$ is coherent. Q.E.D.

Corollary. *Suppose S is a subvariety of dimension ρ in a complex space $(X, \mathscr{H})$ and $\theta: X - S \to X$ is the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on $X - S$ such that the homological codimension* (p. 358, [9]) *of the $\mathscr{H}_x$-module $\mathscr{F}_x \geqq \rho+2$ for $x \in X$. Then the following conditions are equivalent*:

(i) *$R^0\theta(\mathscr{F})$ is coherent.*

(ii) *There exists a coherent analytic sheaf on X which extends $\mathscr{F}$.*

(iii) *$\mathscr{F}$ satisfies $(*)_{X,S}$.*

Proof. Follows from Theorem 2 and Satz I [9]. Q.E.D.

Remark. [14, (4.1)] is a special case of the Corollary to Theorem 2.

III. **Extensions of global sections of coherent sheaves.**

Definition 4. Suppose ρ is a natural number. A real-valued function v on a complex space X is said to be *$*$-strongly ρ-convex* at $x \in X$ if there exist a nowhere degenerate holomorphic map φ from some open neighborhood U of x in X to an open subset D of C^n and a real-valued C^2 function $\tilde{v}$ on D such that $v = \tilde{v}\varphi$ on U and at every point in D the Hermitian matrix $(\partial^2\tilde{v}/\partial z_i\, \partial\bar{z}_j)_{1 \leqq i,j \leqq n}$ has at least $n-\rho+1$ positive eigenvalues.

Definition 5. Suppose ρ is a natural number. An open subset D of a complex space X is said to be *$*$-strongly ρ-concave* at $x \in X$ if there is a *$*$-strongly ρ-convex* function v on some open neighborhood U of x in X such that $D \cap U = \{y \in U \mid v(y) > v(x)\}$.

LEMMA 12. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a reduced complex space $(X, \mathcal{O})$ of pure dimension n such that $E^{n-1}(0, \mathscr{F}) = \varnothing$. Suppose $1 \leqq \rho < n$, $x \in X$, and D is an open subset of X which is *-strongly ρ-concave at x. Then there exist an open neighborhood U of x in X, a subvariety V of dimension $< \rho$ in U, and a natural number m satisfying the following: If for some open neighborhood W of x in U $f \in \Gamma(W, \mathcal{O})$ vanishes identically on $V \cap W$ and $s \in \Gamma(W \cap D, \mathscr{F})$, then $f^m s \mid W' \cap D$ can be extended to an element of $\Gamma(W', \mathscr{F})$ for some open neighborhood W' of x in W.*

Proof. Let $\pi: (\tilde{X}, \tilde{\mathcal{O}}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\tilde{\mathscr{F}}$ be the inverse image of $\mathscr{F}$ under π, $\mathscr{T}$ be the torsion subsheaf of $\tilde{\mathscr{F}}$, and $\mathscr{G} = \tilde{\mathscr{F}}/\mathscr{T}$. Let $\pi^{-1}(x) = (y_1, \ldots, y_k)$. For every $1 \leqq i \leqq k$ there exists a sheaf-monomorphism $\alpha_i: \mathscr{G} \to \tilde{\mathcal{O}}^{p_i}$ on some open neighborhood U_i of y_i in X. By shrinking U_i, $1 \leqq i \leqq k$, we can suppose that $U_i \cap U_j = \varnothing$ for $i \neq j$. There is an open neighborhood U^* of x in X such that $\pi^{-1}(U^*) \subset \bigcup_{i=1}^k U_i$. Define a coherent analytic sheaf $\mathscr{S}$ on $\pi^{-1}(U^*)$ by setting $\mathscr{S} = \tilde{\mathcal{O}}^{p_i}$ on $\pi^{-1}(U^*) \cap U_i$ for $1 \leqq i \leqq k$. Define $\alpha: \mathscr{G} \to \mathscr{S}$ on $\pi^{-1}(U^*)$ by setting $\alpha = \alpha_i$ on $\pi^{-1}(U^*) \cap U_i$ for $1 \leqq i \leqq k$. Let $\beta: R^0\pi(\tilde{\mathscr{F}}) \to R^0\pi(\mathscr{G})$ and $\gamma: R^0\pi(\mathscr{G}) \to R^0\pi(\mathscr{S})$ on U^* be induced respectively by the quotient map $\tilde{\mathscr{F}} \to \mathscr{G}$ and α. Let $\lambda: \mathscr{F} \to R^0\pi(\tilde{\mathscr{F}})$ be the natural map. $E^{n-1}(0, \mathscr{F}) = \varnothing$ implies that $\xi = \gamma\beta\lambda: \mathscr{F} \to R^0\pi(\mathscr{S})$ on U^* is injective. Let $V^* = E^{\rho-1}(\xi(\mathscr{F}), R^0\pi(\mathscr{S}))$ and let $\mathscr{I}$ be the ideal-sheaf on U^* for V^*. By Proposition 1 dim $V^* < \rho$. Let $\mathscr{A} = \xi(\mathscr{F}) : \xi(\mathscr{F})_{[\rho-1]R^0\pi(\mathscr{S})}$. Then $E(\mathscr{A}, \mathcal{O} \mid U^*) = V^*$. Let U be a relatively compact open neighborhood of x in U^*. By Hilbert Nullstellensatz there is a natural number m such that $\mathscr{I}^m \subset \mathscr{A}$ on U. Let $V = V^* \cap U$. We claim that U, V and m satisfy the requirements.

Suppose for some open neighborhood W of x in U we have $f \in \Gamma(W, \mathcal{O})$ vanishing identically on $V \cap W$ and $s \in \Gamma(W \cap D, \mathscr{F})$. By Proposition 6.1, [3], for some open neighborhood W' of x in W $\xi(s) \mid W' \cap D$ can be extended to $t \in \Gamma(W', R^0\pi(\mathscr{S}))$. Let $Z = \{y \in W' \mid t_y \notin \xi(\mathscr{F})_y\}$. $Z = E((\xi(\mathscr{F}) : \mathcal{O}t), \mathcal{O} \mid W')$ is a subvariety in W'. Since D is *-strongly ρ-concave at x, every subvariety-germ of dimension $\geqq \rho$ at x intersects D (4^0 of Definition 2.8 and Proposition 2.9, [3]). Hence $Z \cap D = \varnothing$ implies that dim $Z_x < \rho$. By shrinking W', we can assume that dim $Z < \rho$.

$$t \in \Gamma(W', \xi(\mathscr{F})_{[\rho-1]R^0\pi(\mathscr{S})}).$$

$f^m t \in \Gamma(W', \xi(\mathscr{F}))$. $\xi^{-1}(f^m t) \in \Gamma(W', \mathscr{F})$ extends $f^m s \mid W' \cap D$. Q.E.D.

LEMMA 13. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ of pure dimension n such that $E^{n-1}(0, \mathscr{F}) = \varnothing$. Suppose $1 \leqq \rho < n$, $x \in X$, and D is an open subset of X which is *-strongly ρ-concave at x. Then there exist an open neighborhood U of x in X, a subvariety V of dimension $< \rho$ in U, and a natural number m satisfying the following: If for some open neighborhood W of x in U $f \in \Gamma(W, \mathscr{H})$*

vanishes identically on $V \cap W$ *and* $s \in \Gamma(W \cap D, \mathscr{F})$, *then* $f^m s \mid W' \cap D$ *can be extended to an element of* $\Gamma(W', \mathscr{F})$ *for some open neighborhood* W' *of* x *in* W.

Proof. Let $\mathscr{K}$ be the subsheaf of all nilpotent elements of $\mathscr{H}$ and $\mathscr{O}=\mathscr{H}/\mathscr{K}$. Since the Lemma is local in nature, we can suppose that $\mathscr{K}^k=0$ for some natural number k. For $0 \leqq l \leqq k$ define $\mathscr{F}^{(l)}$ inductively as follows:

$$\mathscr{F}^{(0)} = \mathscr{F}, \quad \text{and, for } 1 \leqq l \leqq k, \quad \mathscr{F}^{(l)} = (\mathscr{K}\mathscr{F}^{(l-1)})_{[n-1]\mathscr{F}^{(l-1)}}.$$

As in the Proof of Lemma 5, we have the following:

$$\mathscr{F}^{(k)} = 0; \qquad E^{n-1}(0, \mathscr{F}^{(l-1)}/\mathscr{F}^{(l)}) = \varnothing \quad \text{for } 1 \leqq l \leqq k;$$

and $\mathscr{G}^{(l)}=\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)}$, $0 \leqq l \leqq k-1$, can be regarded as a coherent analytic sheaf on the reduced complex space $(X, \mathscr{O})$. By Lemma 12 for $0 \leqq l \leqq k-1$ we have a subvariety V_l of dimension $<\rho$ in some open neighborhood U_l of x in X and a natural number p_l satisfying the following: If for some open neighborhood W of x in U_l $f \in \Gamma(W, \mathscr{O})$ vanishes identically on $V_l \cap W$ and $s \in \Gamma(W \cap D, \mathscr{G}^{(l)})$, then $f^{p_l} s \mid W' \cap D$ can be extended to an element of $\Gamma(W', \mathscr{G}^{(l)})$ for some open neighborhood W' of x in W.

Let $U=\bigcap_{l=0}^{k-1} U_l$ and $V=\bigcup_{l=0}^{k-1} (V_l \cap U)$. Let $m_l=\sum_{i=l}^{k-1} p_i$, $0 \leqq l \leqq k-1$. By considering the exact sequences $0 \to \mathscr{F}^{(l+1)} \to \mathscr{F}^{(l)} \to \mathscr{G}^{(l)} \to 0$, $0 \leqq l \leqq k-1$, and by backward induction on l, we conclude the following for $0 \leqq l \leqq k-1$: If $f \in \Gamma(W, \mathscr{H})$ vanishes identically on $W \cap V$ and $s \in \Gamma(W \cap D, \mathscr{F}^{(l)})$ for some open neighborhood W of x in U, then $f^{m_l} s \mid W' \cap D$ can be extended to an element of $\Gamma(W', \mathscr{F}^{(l)})$ for some open neighborhood W' of x in W. Hence U, V, and $m=m_0$ satisfy the requirements. Q.E.D.

LEMMA 14. *Suppose* $\mathscr{F}$ *is a coherent analytic sheaf on a complex space* $(X, \mathscr{H})$ *and* ρ *is a natural number such that* $E^\rho(0, \mathscr{F})=\varnothing$. *Suppose* $x \in X$ *and* D *is an open subset of* X *which is *-strongly* ρ*-concave at* x. *Then there exist an open neighborhood* U *of* x *in* X, *a subvariety* V *of dimension* $<\rho$ *in* U, *and a natural number* m *satisfying the following: If for some open neighborhood* W *of* x *in* U $f \in \Gamma(W, \mathscr{H})$ *vanishes identically on* $W \cap V$ *and* $s \in \Gamma(W \cap D, \mathscr{F})$, *then* $f^m s \mid W' \cap D$ *can be extended to an element of* $\Gamma(W', \mathscr{F})$ *for some neighborhood* W' *of* x *in* W.

Proof. Since the problem is local in nature, we can suppose that X is of finite dimension n. If $n \leqq \rho$, $E^\rho(0, \mathscr{F})=\varnothing$ implies that $\mathscr{F}=0$ and what is to be proved is trivial. So we can suppose that $n>\rho$. Define $\mathscr{G}^{(k)}=0_{[k]\mathscr{F}}$ for $\rho \leqq k \leqq n$. $\mathscr{G}^{(\rho)}=0$. For $\rho<k \leqq n$ let $X_k=\operatorname{Supp} \mathscr{G}^{(k)}/\mathscr{G}^{(k-1)}$ and let $\mathscr{A}^{(k)}$ be the annihilator-ideal-sheaf for $\mathscr{G}^{(k)}/\mathscr{G}^{(k-1)}$. For $\rho<k \leqq n$ X_k is empty or of pure dimension k, $E^{k-1}(0, \mathscr{G}^{(k)}/\mathscr{G}^{(k-1)}) =\varnothing$, and $(\mathscr{G}^{(k)}/\mathscr{G}^{(k-1)}) \mid X_k$ can be regarded as a coherent analytic sheaf on the complex space $(X_k, (\mathscr{H}/\mathscr{A}^{(k)}) \mid X_k)$. By Lemma 13, for $\rho<k \leqq n$, if $x \in X_k$, there exist a subvariety V_k of dimension $<\rho$ in some open neighborhood U_k of x in X_k and a natural number p_k satisfying the following: If for some open neighborhood W of x in U_k $f \in \Gamma(W, (\mathscr{H}/\mathscr{A}^{(k)})|X_k)$ vanishes identically on $W \cap V_k$ and

$$s \in \Gamma(W \cap D, \mathscr{G}^{(k)}/\mathscr{G}^{(k-1)}),$$

then $f^{p_k}s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathscr{G}^{(k)}/\mathscr{G}^{(k-1)})$ for some open neighborhood W' of x in W. For $\rho < k \leqq n$, if $x \in X_k$, choose an open neighborhood $\tilde{U}_k$ of x in X such that $\tilde{U}_k \cap X_k = U_k$; and, if $x \notin X_k$, let $\tilde{U}_k = X$, $V_k = \varnothing$, and $p_k = 1$.

Let $U = \bigcap_{k=\rho+1}^{n} \tilde{U}_k$ and $V = \bigcup_{k=\rho+1}^{n} (U \cap V_k)$. Set $m_k = \sum_{i=\rho+1}^{k} p_i$. By considering the exact sequences $0 \to \mathscr{G}^{(k)} \to \mathscr{G}^{(k+1)} \to \mathscr{G}^{(k+1)}/\mathscr{G}^{(k)} \to 0$, $\rho \leqq k \leqq n-1$, and by induction on k, we conclude the following for $\rho < k \leqq n$: If for some open neighborhood W of x in U $f \in \Gamma(W, \mathscr{H})$ vanishes on $V \cap W$ and $s \in \Gamma(W \cap D, \mathscr{G}^{(k)})$, then $f^{m_k}s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathscr{G}^{(k)})$ for some open neighborhood W' of x in W. The Lemma follows from $\mathscr{F} = \mathscr{G}^{(n)}$ and $m = m_n$. Q.E.D.

THEOREM 3 (LOCAL EXTENSION). *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ and ρ is a natural number such that $\mathscr{F} = \mathscr{F}^{[\rho-1]}$. Suppose $x \in X$ and D is an open subset of X which is *-strongly ρ-concave at x. Then the following is satisfied: If $s \in \Gamma(W \cap D, \mathscr{F})$ for some open neighborhood W of x in X, then $s|W' \cap D$ can be extended to an element t of $\Gamma(W', \mathscr{F})$ for some open neighborhood W' of x in W and t_x is uniquely determined.*

Proof. Since $\mathscr{F} = \mathscr{F}^{[\rho-1]}$, by Theorem 1, and the definition of $\mathscr{F}^{[\rho-1]}$, $E^\rho(0, \mathscr{F}) = \varnothing$. There exist an open neighborhood U of x in X, a subvariety V of dimension $< \rho$ in U, and a natural number m satisfying the requirements of Lemma 14. By Lemma 3 every branch of $E^\sigma(0, \mathscr{F})$ has dimension $> \rho$ for every nonnegative integer σ. By shrinking U we can assume that there is $f \in \Gamma(U, \mathscr{H})$ such that f vanishes identically on V and f does not vanish identically on any branch of $E^\sigma(0, \mathscr{F}) \cap U$ for any nonnegative integer σ. By Lemma 2 the sheaf-homomorphism $\alpha: \mathscr{F} \to \mathscr{F}$ on U defined by multiplication by f^m is injective.

Suppose $s \in \Gamma(W \cap D, \mathscr{F})$. For some open neighborhood W' of x in W $\alpha(s)|W' \cap D = f^m s|W' \cap D$ can be extended to an element $\tilde{t} \in \Gamma(W', \mathscr{F})$. $Z = \{y \in W' \mid \tilde{t}_y \notin \alpha(\mathscr{F})_y\}$ is a subvariety in W'. Since D is *-strongly ρ-concave at x and $Z \cap D = \varnothing$, either $x \notin Z$ or $\dim Z_x < \rho$. By shrinking W', we can assume that either $Z \cap W' = \varnothing$ or $\dim Z < \rho$. $\tilde{t} \in \Gamma(W', \alpha(\mathscr{F})_{[\rho-1]\mathscr{F}})$. $\mathscr{F} = \mathscr{F}^{[\rho-1]}$ implies that $\alpha(\mathscr{F})_{[\rho-1]\mathscr{F}} = \alpha(\mathscr{F})$. Hence $\tilde{t} \in \Gamma(W', \alpha(\mathscr{F}))$. $t = \alpha^{-1}(\tilde{t}) \in \Gamma(W', \mathscr{F})$ extends $s|W' \cap D$.

Suppose for some other open neighborhood W'' of x in W there is $t' \in \Gamma(W'', \mathscr{F})$ extending $s|W'' \cap D$. We are going to prove that $t'_x = t_x$. By shrinking both W' and W'', we can assume that $W' = W''$. $Y = \{y \in W' \mid t'_y \neq t_y\}$ is a subvariety in W'. Since D is *-strongly ρ-concave at x and $Y \cap D = \varnothing$, either $x \notin Y$ or $t'_x - t_x \in (0_{[\rho-1]\mathscr{F}})_x = 0$. Q.E.D.

THEOREM 4 (GLOBAL EXTENSION). *Suppose ρ is a natural number and v is a *-strongly ρ-convex function on a complex space X such that $\{x\in X \mid \lambda<v(x)<\mu\}$ is relatively compact in X for any two real numbers $\lambda<\mu$. Suppose $\mathscr{F}$ is a coherent analytic sheaf on X satisfying $\mathscr{F}=\mathscr{F}^{[\rho-1]}$. Then for $\lambda\in \boldsymbol{R}$ every section of $\mathscr{F}$ on $X_\lambda=\{x\in X \mid v(x)>\lambda\}$ is uniquely extendible to a section of $\mathscr{F}$ on X.*

Proof. We can assume that X as a topological space is connected. Since $E^\rho(0, \mathscr{F}) = \varnothing$, we can assume that every branch of X has dimension $>\rho$. Fix $\lambda_0\in\boldsymbol{R}$ and $s\in\Gamma(X_{\lambda_0}, \mathscr{F})$. We can assume that $X_{\lambda_0}\neq\varnothing$. Let $\Lambda=\{\lambda\in\boldsymbol{R} \mid \lambda\leqq\lambda_0$ and s can be extended to $s_\lambda\in\Gamma(X_\lambda, \mathscr{F})\}$. Clearly, if $\lambda\in\Lambda$ and $\lambda<\mu$, then $\mu\in\Lambda$. We are going to prove:

(6) $$\text{If } \lambda\in\Lambda \text{ and } s_\lambda, s'_\lambda\in\Gamma(X_\lambda, \mathscr{F}) \text{ both extend } s, \text{ then } s_\lambda = s'_\lambda.$$

Suppose the contrary. Then $Z=\{x\in X_\lambda \mid (s_\lambda)_x\neq(s'_\lambda)_x\}$ is a nonempty subvariety in X_λ. Let Z_0 be a branch of Z. Take $x^*\in Z_0$ and let $\lambda^*=v(x^*)$. Let $\xi=\sup\{v(x) \mid x\in Z_0\}$. Since $Z\cap X_{\lambda_0}=\varnothing$, ξ is the supremum of v on the compact set $Z_0\cap\{x\in X \mid \lambda^*\leqq v(x)\leqq\lambda_0\}$. $\xi=v(y)$ for some $y\in Z_0$. Since X_ξ is *-strongly ρ-concave at y and $Z_0\cap X_\xi=\varnothing$, we have $\dim(Z_0)_y<\rho$. Since Z_0 is irreducible, $\dim Z_0<\rho$. Hence $\dim Z<\rho$. $s_\lambda-s'_\lambda\in\Gamma(X_\lambda, 0_{[\rho-1]\mathscr{F}})$. (6) follows from $0_{[\rho-1]\mathscr{F}}=0$.

For $\lambda\in\Lambda$ denote the unique element of $\Gamma(X_\lambda, \mathscr{F})$ which extends s by s_λ. To finish the proof, we need only prove that Λ has no lower bound, because in that case $\Lambda=\{\lambda\in\boldsymbol{R} \mid \lambda\leqq\lambda_0\}$ and by (6) $s^*\in\Gamma(X, \mathscr{F})$ defined by $s^*|X_\lambda=s_\lambda$ for $\lambda\in\Lambda$ extends s. Suppose the contrary. Then $\eta=\inf\Lambda$ exists and is finite. Since X is connected, this implies that X_η is not closed in X. By Theorem 3 for every x in the boundary ∂X_η of X_η there exists an open neighborhood U_x of x in X such that s_η can be extended to $t_{(x)}\in\Gamma(U_x\cup X_\eta, \mathscr{F})$. For $x, x'\in\partial X_\eta$ let $Y_{(x,x')}=\{z\in U_x\cap U_{x'} \mid (t_{(x)})_z\neq(t_{(x')})_z\}$. Since $0_{[\rho-1]\mathscr{F}}=\varnothing$, $Y_{(x,x')}$ is either empty or every branch of $Y_{(x,x')}$ has dimension $\geqq\rho$. Since X_η is *-strongly ρ-concave at every one of its boundary points,

(7) $$Y_{(x,x')}\cap\partial X_\eta = \varnothing \quad \text{for } x, x'\in\partial X_\eta.$$

Since ∂X_η is compact we can choose $x_1,\dots,x_k\in\partial X_\eta$ such that $\partial X_\eta\subset\bigcup_{i=1}^k U_{x_i}$. For $1\leqq i\leqq k$ choose a relatively compact open neighborhood W_i of x_i in U_{x_i} such that $\partial X_\eta\subset\bigcup_{i=1}^k W_i$. Let W_i^- be the closure of W_i in X, $1\leqq i\leqq k$. (7) implies that we can choose an open neighborhood W of ∂X_η in $\bigcup_{i=1}^k W_i$ such that W does not intersect the closed set $\bigcup_{1\leqq i,j\leqq k, i\neq j} Y_{(x_i,x_j)}\cap W_i^-\cap W_j^-$. For some $\lambda<\eta$, $X_\lambda\subset W\cup X_\eta$ because of Proposition 2.7 of [3]. Define $t\in\Gamma(X_\lambda, \mathscr{F})$ by setting $t=s_{(x_i)}$ on $(U_{x_i}\cup X_\eta)\cap X_\lambda$. t extends s, contradicting $\lambda\notin\Lambda$.

Uniqueness follows from (6). Q.E.D.

REMARKS. (i) Theorem 3 generalizes the Theorem on p. 279 of [4] and Theorem 4 generalizes Corollary 5.2 of [4] because of Theorem 4.3 of [4]. Theorems 3 and 4 here have the advantage that, if $\mathscr{F}$ does not satisfy $\mathscr{F}=\mathscr{F}^{[\rho-1]}$, we can always construct the coherent analytic sheaf $\mathscr{G}=(\mathscr{F}/0_{[\rho]\mathscr{F}})^{[\rho-1]}$ which satisfies $\mathscr{G}=\mathscr{G}^{[\rho-1]}$.

(ii) Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex space $(X, \mathscr{H})$ and $x \in X$. The condition $\mathscr{F}_x=(\mathscr{F}^{[0]})_x$ is equivalent to the condition codh $\mathscr{F}_x \geqq 2$. It can be proved in the following way: If $\mathscr{F}_x=(\mathscr{F}^{[0]})_x$, then $E^0(0, \mathscr{F})=\varnothing$ and by Lemmas 2 and 3 we can find $f \in \Gamma(U, \mathscr{H})$ for some open neighborhood U of x in X such that f_x is not a unit of $\mathscr{H}_x$ and f_x is not a zero-divisor for $\mathscr{F}_x$. By shrinking U, we can assume that f_y is not a zero-divisor for $\mathscr{F}_y$ for $y \in U$. Suppose $x \in E^0(f\mathscr{F}, \mathscr{F} \mid U)$. By shrinking U, we can find $g \in \Gamma(U, \mathscr{F})$ such that $g_y \in (f\mathscr{F})_y$ for $y \in U-\{x\}$ and $g_x \notin (f\mathscr{F})_x$. Then $h \in \Gamma(U, \mathscr{F}^{[0]})$ defined by $g_y=f_y h_y$ for $y \in U-\{x\}$ does not satisfy $h_x \in \mathscr{F}_x$. This is a contradiction. Hence $x \notin E^0(f\mathscr{F}, \mathscr{F} \mid U)$. By Lemmas 2 and 3 we can find $s \in \mathscr{H}_x$ which vanishes at x and is not a zero-divisor for $(\mathscr{F}/f\mathscr{F})_x$. codh $\mathscr{F}_x \geqq 2$. On the other hand codh $\mathscr{F}_x \geqq 2$ implies $\mathscr{F}_x=(\mathscr{F}^{[0]})_x$ by Korollar zu Satz III, [9].

The equivalence of $\mathscr{F}_x=(\mathscr{F}^{[0]})_x$ and codh $\mathscr{F}_x \geqq 2$ is also a consequence of [14, (1.1)]. However, the proof presented here is more conceptual than the proof in [14].

(iii) In the case of Stein spaces we have the following stronger version ofTheorem 4 which generalizes Theorem 5.4 of [4]:

(8) Suppose $\mathscr{F}$ is a coherent analytic sheaf on a Stein space X such that $\mathscr{F}=\mathscr{F}^{[0]}$. Suppose K is a compact subset of X such that, if A is a branch of $E^\sigma(0, \mathscr{F})$ for any $\sigma \geqq 2$, then $A-K$ is irreducible. Then for every open neighborhood U of K in X every element of $\Gamma(U-K, \mathscr{F})$ can be extended uniquely to an element of $\Gamma(U, \mathscr{F})$.

It can be proved in the following way: Suppose $s \in \Gamma(U-K, \mathscr{F})$. Since $H^1(X, \mathscr{F})=0$, from the Mayer-Vietoris sequence of $\mathscr{F}$ on $X=(X-K) \cup U$ (p. 236, [2]) we conclude that for some $f \in \Gamma(X-K, \mathscr{F})$ and $g \in \Gamma(U, \mathscr{F})$ $f-g=s$ on $U-K$. From Theorem 4 we can find $\tilde{f} \in (X, \mathscr{F})$ which agrees with f outside some compact subset of X. Since $E^\sigma(0, \mathscr{F})=\varnothing$ for $\sigma \leqq 1$ and $A-K$ is irreducible for any branch A of $E^\sigma(0, \mathscr{F})$ with $\sigma \geqq 2$, f agrees with $\tilde{f}$ on $X-K$. $(\tilde{f} \mid U)-g$ extends s. The extension is clearly unique, because $E^0(0, \mathscr{F})=\varnothing$.

In view of the equivalence of $\mathscr{F}_x=(\mathscr{F}^{[0]})_x$ and codh $\mathscr{F}_x \geqq 2$, in the above proof we can use Theorem 15 of [2] instead of Theorem 4. So (8) can be proved also by the finiteness theorems of pseudoconvex spaces in [2].

(8) generalizes Theorem 5.4 of [4] because of the following:

(9) Suppose K is a closed subset of an irreducible complex space X and U is an open neighborhood of K in X such that for every branch A of U $A-K$ is irreducible. Then $X-K$ is irreducible.

Let R be the set of all regular points of X. To prove (9), we need only show that $R-K$ is connected. Suppose $R \cap U = \bigcup_{i \in I} R_i$ is the decomposition into topological components. Then $R_i - K$ is connected for $i \in I$. The restriction map $\Gamma(R \cap U, C) \to \Gamma(R \cap (U-K), C)$ is an isomorphism. From the following portion of the Mayer-Vietoris sequence of the constant sheaf C on $R = (R \cap U) \cup (R-K)$: $0 \to \Gamma(R, C) \to \Gamma(R-K, C) \oplus \Gamma(R \cap U, C) \to \Gamma(R \cap (U-K), C)$, we conclude that the restriction map $\Gamma(R, C) \to \Gamma(R-K, C)$ is an isomorphism. $R-K$ is connected.

References

1. A. Andreotti, *Théorèmes de dépendance algébrique sur les espaces complexes pseudoconcaves*, Bull. Soc. Math. France **91** (1963), 1–38.

2. A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

3. H. Fujimoto, *On the continuation of analytic sets*, J. Math. Soc. Japan **18** (1966), 51–85.

4. ———, *The continuation of sections of torsion-free coherent analytic sheaves*, Nagoya Math. J. **31** (1968), 279–294.

5. H. Grauert, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Inst. Hautes Études Sci. Publ. Math. No. 5, 1960.

6. H. Grauert and R. Remmert, *Bilder und Urbilder analytischer Garben*, Ann. of Math. **68** (1958), 393–443.

7. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

8. M. Nagata, *Local rings*, Interscience, New York, 1962.

9. G. Scheja, *Riemannsche Hebbarkeitssätze für Cohomologie Klassen*, Math. Ann. **144** (1961), 345–360.

10. ———, *Fortsetzungssätze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung*, Math. Ann. **157** (1964), 75–94.

11. J.-P. Serre, *Prolongement de faisceaux analytiques cohérents*, Ann. Inst. Fourier (Grenoble) **16** (1966), 363–374.

12. Y.-T. Siu, *Noether-Lasker decomposition of coherent analytic subsheaves*, Trans. Amer. Math. Soc. **135** (1969), 375–384.

13. W. Thimm, *Lückengarben von kohärenten analytischen Modulgarben*, Math. Ann. **148** (1962), 372–394.

14. G. Trautmann, *Ein Kontinuitätssatz für die Fortsetzung kohärenter analytischen Garben*, Arch. Math. (Basel) **18** (1967), 188–196.

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